12. THE AXIOM OF FOUNDATION

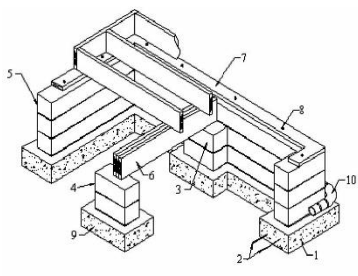
§12.1. Explanation of the Axiom

Can a set be an element of itself? It does rather seem counter-intuitive. If we'd allowed the existence of the set of all sets then this would certainly be possible. But, as we've seen, the set of all sets leads to the Russell Paradox, and we've constructed the ZF axioms to rule this out. Still, might there not be sets S where $S \in S$?

There's an unhealthy recursiveness in having $S \in S$ and so we'd probably want to rule it out. So we add the following axiom, sometimes called the Axiom of Foundation, to our ZF axioms. This says that every non-empty set contains an element that is disjoint from it.

Axiom of Foundation (AF): Every non-empty set contains an element that is disjoint from itself. That is, if S is a set then there exists T such that:

$$T \in S \text{ and } S \cap T = \varnothing.$$



Theorem 1: If x is a set in a set theory that satisfies ZF + AF then $x \notin x$.

Proof: Suppose $x \in x$ and let $S = \{x\}$.

By the Axiom of Foundation S contains an element T such that $S \cap T = \varnothing$.
Clearly it must be that $T = x$, so $x \in S \cap T$, a contradiction.
✋☺

Ideally we'd like to prove that the Axiom of Foundation follows from the ZF axioms, possibly supplemented by the Axiom of Choice. But this is not possible. The Axiom of Foundation has been shown to be consistent with, but independent from, the ZF axioms, with or without the Axiom of Choice.

No theorem in mainstream mathematics requires the Axiom of Foundation. It would have made a couple of proofs regarding ordinal numbers a little easier to prove. Unlike the Axiom of Choice it doesn't have the same usefulness in simplifying statements of theorems and so my preference is to be agnostic – that is, to neither assume the Axiom of Foundation or to deny it.

In fact it has been shown that there are at least eight different set theories on which we could build mathematics:

ZF	ZF + AXC	ZF + CH	ZF + AXC + CH
ZF + AF	ZF + AXC + AF	ZF + CH + AF	**ZF + AXC + CH + AF**

All are consistent, assuming that the ZF axioms on their own are consistent. My personal choice is:

ZF + AXC + CH + AF.

I've explained why I like to 'believe' in the Axiom of Choice and the Continuum Hypothesis.

A similar argument applies to the Axiom of Foundation. If there is a set that is an element of itself then we'll never be able to explicitly describe it.

§12.2. Consequences of the Axiom of Foundation

Just as counter-intuitive as having a set that is an element of itself is to have two sets that are elements of one another, or a descending sequence of sets, each containing the next. The Axiom of Foundation rules out both.

Theorem 2: If S, T are sets in a set theory that satisfies ZF + AF and $S \in T$ then $T \notin S$.
Proof: If $S \in T$ and $T \in S$ then {S, T} does not contain an element that is disjoint from itself. ✋☺

Theorem 3: In a set theory that satisfies ZF + AF there is no infinite sequence of sets $S_1, S_2, \ldots$, such that

$$\ldots\ldots \in S_3 \in S_2 \in S_1 \in S_0.$$

Proof: Apply the Axiom of Foundation to the set
$S = \{S_0, S_1, S_2, \ldots\}$. ✋☺

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- Concepts of Calculus

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- Techniques of Algebra
- Techniques of Calculus
- Matrices

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- Linear Algebra
- Languages & Machines
- Discrete Mathematics

3rd YEAR UNI
- Group Theory vol 1
- Group Theory vol 2
- Galois Theory
- Geometry vol 1
- Geometry vol 2
- Topology
- Set Theory
- Number Theory
- Graph Theory
- Complex Variables

POSTGRADUATE
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- Representation Theory
- Quadratic Forms
- Group Tables vol 1
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